

Functional Solution Scheme for Relativistic Strong Coupling Theory I

By G. HEBER * and H. J. KAISER **

Joint Institute for Nuclear Research, Laboratory of Theoretical Physics, Dubna (Moscow)

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The vacuum expectation value of the S -matrix is represented, following Hori, as a functional integral and separated according to $S_{\text{vac}} = \exp(-iW) \int D\varphi \exp(-i \int dx L_w)$. Now, the functional integral involves only the part L_w of the Lagrangian without derivatives and can be easily calculated in lattice space. We propose a graphical scheme which formalizes the action of the operator $W = \int dx dy \delta(x-y) (\delta/\delta\varphi(y)) \square_x (\delta/\delta\varphi(x))$. The scheme is worked out in some detail for the calculation of the two-point-function of neutral BOSE fields with the self-interaction $\lambda \varphi^M$ for even M . A method is proposed which under certain convergence assumptions should yield in a finite number of steps the lowest mass eigenvalues and the related matrix elements. The method exhibits characteristic differences between renormalizable and nonrenormalizable theories.

The aim of this paper is to present an approximation method for the practical calculation of the 2-point-function of a quantum system which is defined by its Lagrangian. The method should work especially well for strong interactions (or strong nonlinearities) and should be LORENTZ invariant. We started from Hori's method¹. The main idea of Hori is the following: The 2-point-function can be represented formally as

$$\chi(x_1, x_2) = -S_{\text{vac}}^{-1} \frac{\delta^2 S_{\text{vac}}}{\delta\varphi(x_1) \delta\varphi(x_2)} \Big|_{\varphi=0} \quad (1)$$

with

$$S_{\text{vac}} = \langle T \exp(i \int dx L_w) \rangle_{\text{vac}}^2$$

S_{vac} can be written in the form of a functional integral

$$S_{\text{vac}} = \int D\varphi \exp(i \int dx L) \quad (2)$$

where φ is the classical field quantity of the system. The integral (2) could be evaluated in lattice-space if L would not contain derivatives of φ . Therefore, Hori proposes the decomposition

$$S_{\text{vac}} = e^{-iW} \Omega \quad (3)$$

with

$$\Omega = \int D\varphi \exp(i \int dx L_w) \quad (4)$$

where L_w is the part of L without derivatives, so Ω can be calculated in lattice-space. The operator W stems from the difference of L and L_w . If we have^{2a}

e. g. $L - L_w = \varphi(x) \square \varphi(x)$ it follows

$$W = \int dx dy \delta(x-y) \frac{\delta}{\delta\varphi(y)} \square_x \frac{\delta}{\delta\varphi(x)}. \quad (5)$$

In order to calculate S_{vac} from (3), (4) and (5) in practice one of us proposed³ to expand S_{vac} and Ω into VOLTERRA-series with respect to the external source:

$$S_{\text{vac}}[\varphi] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n \cdot T_n(x_1 \dots x_n) e_n(x_1 \dots x_n), \quad (6)$$

$$\Omega[\varphi] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n \cdot \varphi_n(x_1 \dots x_n) e_n(x_1 \dots x_n) \quad (7)$$

with⁴

$$e_n(x_1 \dots x_n) = \varphi(x_1) \dots \varphi(x_n).$$

We then have⁵

$$\chi(x_1, x_2) = -T_2(x_1, x_2)/T_0,$$

$$T_2(x_1, x_2) = \sum_{n=2, 4, 6, \dots}^{\infty} \frac{i^{(n-2)/2}}{[(n-2)/2]!} \int dx_3 \dots dx_n \cdot G(3, 4) G(5, 6) \dots G(n-1, n) \varphi_n(x_1 \dots x_n) \quad (8)$$

where

$$G(i, j) = \delta(x_i - x_j) \square_j.$$

The main difficulty in evaluating (8) was (see³) that one must sum the whole infinite series (8)

* On leave from Universität Leipzig, DDR.

** On leave from Technische Universität Dresden, DDR.

¹ S. Hori, Nucl. Phys. **30**, 644 [1962].

² L is the Lagrangian of the system including the interaction with the external source $\varphi(x)$.

^{2a} The usual factor $\frac{1}{2}$ is here left out, its addition would imply only minor changes (e. g. in the mass eigenvalues).

³ G. HEBER, Acta Phys. Hungar. **17**, 115 [1964].

⁴ It is important to remark that the functions T_n and φ_n in (6) and (7) are symmetric with respect to all n variables.

⁵ T_0 has similar structure (see Appendix B).



to get a two-point-function which is different from zero for $x_1 \neq x_2$. The aim of this article is to show how to extract some interesting information concerning the two-point-function from the knowledge of a (properly chosen) *finite* number of terms from (8). In order to develop the formalism we shall discuss, as a model case, systems with the Lagrangian

$$L = \varphi(x) \square \varphi(x) + \lambda \varphi^M(x) - \varrho(x) \varphi(x) \quad (9)$$

where λ is a (real) parameter and M is some positive even integer. The variable x may be a vector in either MINKOWSKI-space or in another space of less than four dimensions. For example, we shall find it interesting to discuss also the special case⁶ in which x represents simply the ordinary time t .

1. The φ_n -Functions

The functions φ_n are defined by the integrals

$$\varphi_n(x_1 \dots x_n) = \int D\varphi \varphi(x_1) \dots \varphi(x_n) e^{i \int dx L_w'} \quad (10)$$

where $L_w' = L_w + \varrho \varphi = \lambda \varphi^M(x)$. These integrals can be calculated in lattice space. They reduce to infinite products of integrals of the type

$$\begin{aligned} & \int_{-\infty}^{\infty} dy y^k e^{-i \lambda \varepsilon y^M} = \\ & \begin{cases} \frac{2}{M} (i \lambda \varepsilon)^{-(k+1)/M} \Gamma\left(\frac{k+1}{M}\right) \equiv M_k M_0 & k \text{ even} \neq 0 \\ 0 & k \text{ odd} \end{cases} \quad (11) \\ & M_0 = \int dy e^{-i \lambda \varepsilon y^M} \end{aligned}$$

for even M . ε is the cell volume in the lattice space. The φ_n can now be expressed in the form⁷

$$\varphi_n(x_1 \dots x_n) = \begin{cases} M_n & \text{for } x_n = x_{n-1} = \dots = x_2 = x_1. \\ M_2 M_{n-2} & \text{for } x_n = x_{n-1} \neq x_{n-2} = \dots = x_2 = x_1 \\ & \text{(and permutations)} \\ \vdots \\ M_2^{n/2} & \text{for } x_n = x_{n-1} \neq x_{n-2} = x_{n-3} \neq \dots \\ & \neq x_2 = x_1 \text{ (and permutations)} \\ 0 & \text{for all } x \text{ different or an odd number of} \\ & \text{them coinciding.} \end{cases} \quad (12)$$

⁶ In this case we have, of course, $\square \rightarrow \partial^2/\partial t^2$ and our model is no longer a field theory but simply quantum mechanics.

⁷ The constant φ_0 is without interest; it appears in all T_n and therefore cancels in χ .

⁸ By setting $f_0 = 0$ we have put $\varphi_0 = 1$ (cf. 7).

⁹ We remark that for even M all odd derivatives of f vanish at the point $\varrho = 0$ (see Appendix A). This simplifies con-

siderably the formulae; we shall exploit this fact in the following considerations

$$\Omega[\varrho] = \exp \left\{ \int dx f[\varrho(x)] \right\}. \quad (13)$$

Here $f(\varrho)$ is some function which is determined by the moments M_k (see Appendix A). If we write instead of (13) $\Omega = \sum_{m=0}^{\infty} \frac{1}{m!} [\int dx f(\varrho)]^m$ and introduce the TAYLOR series

$$f(\varrho) = f_0' \varrho + f_0'' \varrho^2/2! + \dots^8,$$

then we get by comparing with (7) directly the φ_n :

$$i^2 \varphi_2(x_1, x_2) = f_0'^2 + f_0'' \delta(x_1 - x_2), \quad (14)$$

$$\begin{aligned} i^4 \varphi_4(x_1 \dots x_4) = & f_0'^4 + f_0'^2 f_0'' [\delta(x_1 - x_2) + \text{permut.}] \\ & + f_0' f_0''' [\delta(x_1 - x_2) \delta(x_2 - x_3) + \text{permut.}] \\ & + f_0''^2 [\delta(x_1 - x_2) \delta(x_3 - x_4) + \text{permut.}] \\ & + f_0'''' \delta(x_1 - x_2) \delta(x_2 - x_3) \delta(x_3 - x_4). \end{aligned} \quad (14a)$$

Similar formulae hold for higher n ⁹. (For proof see Appendix A.) In (14a) we have to add at the indicated places all permutations of the four variables $x_1 \dots x_4$, thus allowing for the coincidence of different points. The φ_n are essentially products of δ -functions. This demands some caution in the evaluation of formulae like (8) in which these δ -functions should be differentiated and then integrated. We have, therefore, controlled all important steps of the calculation by taking the limit of difference quotients. We found the results in agreement with the formal approach of considering the φ_n as differentiable functions. Difficulties appear only if we have to do with products of δ -functions of the same arguments. In all such cases we used the lattice-space-representation of the δ -function, because our functions result from the limiting process "lattice space \rightarrow continuous x -space"¹⁰.

¹⁰ In ref. ³ it was stressed that products of δ -functions are inherently ambiguous. This ambiguity is here lifted by the restriction to the particular representation of all δ -functions in lattice space.

2. Graphical Representation and Evaluation of the Two-Point-Function

The insertion of the φ_n into the series (8) for T_2 yields so many terms that a graphical representation proves very useful for classification and calculation of the various contributions. We introduce the graph scheme by treating as an example the function

$$\delta(x_1 - x_3) \delta(x_4 - x_5) \delta(x_6 - x_7) \dots \delta(x_n - x_2)$$

contributing to φ_n . From (8) we see that we have to calculate

$$\int dx_3 \dots dx_n \delta(x_3 - x_4) \delta(x_5 - x_6) \dots \delta(x_{n-1} - x_n) \cdot \square_4 \square_6 \dots \square_n \delta(x_1 - x_3) \delta(x_4 - x_5) \dots \delta(x_n - x_2). \quad (15)$$

Using the well-known properties of the derivatives of δ -functions (15) becomes

$$\square^{\frac{n-2}{2}} \delta(x_1 - x_2). \quad (15a)$$

To get the graphical representation of (15) we depict temporarily each variable x_i by one point; then we fuse those points which appear in the φ_n -function as arguments of the same δ -function to $n/2$ double-points (see Fig. 1 a). Next we connect by lines all those points which share the same δ -function in the operator \mathcal{W} [eq. (5)] (Fig. 1 b). Finally, we indicate the "free" points 1 and 2 (over which there is no integration) by short lines. Now we can drop the numbers and arrive at Fig. 1 c.

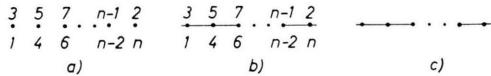


Fig. 1. Steps in deriving the graph scheme.

To illustrate the prescription further, we compile in Figs. 2–4 all terms of T_2 which arise from φ_2 , φ_4 and φ_6 .

$$f^* \delta(12)$$

Fig. 2. Contribution from φ_2 to T_2 .

$$2f^* \square \delta(12) \quad f^{**} \square \delta(0) \delta(12)$$

Fig. 3. Contributions from φ_4 to T_2 .

$$8f^{*3} \square^2 \delta(12) \quad 4f^{**} f^* \square \delta(0) \square \delta(12) \quad 4f^{**} f^* \square \delta(0) \delta(12) \quad f^{(6)} (\square \delta(0))^2 \delta(12) \quad 4f^{**} f^* \square^2 \delta(0) \delta(12)$$

Fig. 4. Contributions from φ_6 to T_2 .

We must add some remarks on these figures.

1) The loop in Figs. 3 b and 4 b, c represents the expression

$$\int dx_3 dx_4 \delta(34) \square_4 \delta(34) \delta(13) \delta(42).$$

2) Similarly the double loop in Fig. 4 d has the value

$$\int dx_3 \dots dx_6 \delta(34) \delta(56) \cdot \square_4 \square_6 \delta(13) \delta(35) \delta(56) \delta(64) \delta(42).$$

3) We have omitted here all graphs which consist of two or more disconnected parts. It can be shown (see Appendix B) that they are vacuum terms which cancel with similar contributions from T_0 [eq. (8)].

4) It is important to note that most of the graphs appear more than once in T_2 . This gives the weight factors to the expressions in Figs. 3 and 4 (see Appendix C). Furthermore, we must multiply by the factor $i^{n/2-1}/(n/2-1)!$ appearing in (8).

5) Terms like $\square \delta(0)$, of course, have only symbolic meaning. They must be always interpreted as limits of the corresponding expression in lattice space (cf. remark at the end of 2). For example, the derivative $(\partial/\partial t)^{2k} \delta(t)$ reads in lattice space

$$\delta^{(2k)}(t) = \frac{(-1)^k}{2^{2k} (\Delta t)^{2k+1}} \begin{cases} (-1)^m \binom{2k}{k+m} & \text{for } t = 2m \Delta t, \\ 0 & \text{else.} \end{cases} \quad (16)$$

Therefore, we have to put

$$\delta^{(2k)}(0) = \lim_{\Delta t \rightarrow 0} \frac{(-1)^k}{2^{2k} (\Delta t)^{2k+1}} \binom{2k}{k}. \quad (16a)$$

The graphical representation, just introduced, has the further advantage of yielding even the results of a good deal of the integrations in (8):

Each line between two points generates one \square .

The number k of lines between the points 1 and 2 therefore, gives the exponent of \square operating in the result on $\delta(12)$. Multiple "bounds" like Fig. 5 a or branchings like Fig. 5 b are easily recognized as $[\square \delta(12)]^3$ and $[\square^2 \delta(12)]^3$ respectively. "Loops" i. e. structures sharing only one point with the skeleton graph (Figs. 3 b, 4 b, c, d) yield constant factors of the type $\square^k \delta(0)$.



Fig. 5. Some graph elements: a) triple bound, b) branching.

Another important aspect of our graphical representation is that we can state a rule for the generation of all terms which contribute to T_2 from a given φ_n (for large n there are very many terms, so such a rule is of great value): Draw first the graph with the highest derivative possible for the given n ¹¹. (The graphs Figs. 1 c, 2, 3 a, 4 a e. g., are of this type.)

Then you can generate all further graphs belonging to the same n by "folding" that basic graph to fuse points in all possible ways. Fig. 6 demonstrates the procedure for the case $n = 6$ ¹² (Fig. 4).

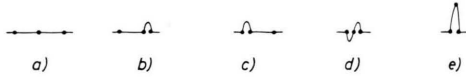


Fig. 6. Construction of the graphs Fig. 4 by fusion of points.

A closer inspection of the contributions to T_2 shows that they differ in their origin (characterized by the index n of φ_n) and that they lead to different powers k of \square operating on $\delta(12)$ ¹³. Therefore, it proves useful to arrange the various contributions to T_2 into a scheme scetched in Fig. 7. Only at the places marked by crosses we have contributions (of course, the figure must be continued to infinity). We remark that all "basic graphs" appear on the diagonal $k = (n - 2)/2$. It must be noted, however, that there exist "splitting" graphs which contribute to more than one point of the scheme Fig. 7 (e. g. graphs containing the elements Fig. 5 a, b). They must be treated by expanding the involved powers of derivatives of δ -functions into a sum of simple derivatives.

	0	1	2	3	4	$\rightarrow k$
2	×					
4	×	×				
6	×	×	×			
8	×	×	×	×		
10	×	×	×	×	×	
$\downarrow n$						

Fig. 7. Arrangement of the contributions (crosses) to T_2 according to order n of graph and power k in result $\square^k \delta(12)$.

Fig. 7 is most important for the discussion of the dependence of the diverse contributions to T_2

on the lattice-constant ε which appears in the functional integrals and expressions like $\square \delta(0)$. It can be proved (see Appendix D) that to each point of Fig. 7 there belongs a quite definite power of ε . Fig. 10 shows an example. We see that the ε -power depends on the interaction exponent M [see eq. (9)] and on the number N of dimensions we consider. Fig. 10 leads to interesting consequences to be discussed in the next paragraph.

3. Properties and Consequences of Our Representation of T_2

It is now clear that we arrive at a representation of T_2 in the form

$$T_2(x_1, x_2) = \sum_{k=0}^{\infty} (-1)^{k+1} \alpha_k \square^k \delta(x_1 - x_2). \quad (17)$$

Here, each α_k is an infinite series of the contributions of the k^{th} vertical of Fig. 7 or 10. The meaning of the series (17) becomes clear after a FOURIER transformation¹⁴:

$$T_2(x) = \int dp \exp \{i p^r x_r\} T_2(p).$$

The result is

$$T_2(p) = - \sum_{k=0}^{\infty} \alpha_k (p^2)^k, \quad p^2 \equiv p^r p_r. \quad (17a)$$

That is, we have got an expansion of the FOURIER transform of the two-point-function into powers of p^2 .

At this point it is quite natural to look at the well-known case $M = 2$ (where we do not have any interaction, and the exact T_2 -function is of course $T_2(p) = \text{const}/(p^2 - \lambda)$). It turns out that the sum of the "basic graphs" (which always form the diagonal in Figs. 7 or 10) gives exactly this result. All other contributions to the α_k vanish for $M = 2$ as they are all proportional to some $f_0^{(v)}$ with $v > 2$ and these derivatives are zero because $f(q)$ here simply has the form $f(q) = q^2/4\lambda$.

In all other cases ($M \neq 2$) we have non-zero contributions throughout Fig. 7. Now, the sign of the coefficient $\left(\frac{1}{2} - \frac{1}{N} - \frac{1}{M}\right)$ of n in (D 8) is de-

¹¹ We shall call graphs of this type "basic graphs".

¹² It is, however, not allowed to cut the graph (otherwise one would get vacuum graphs, see Appendix B), and free ends must stay free.

¹³ The l in constants like $\square^l \delta(0)$ is of course not to be counted as contribution to k .

¹⁴ Having calculated all ambiguous products of δ -functions in lattice space we are now free to use any representation of the remaining linear expression (17) in δ -functions. The FOURIER representation seems most suitable.

cisive for the behaviour of the solutions. We have to consider the three cases

$$M \leq 2N/(N-2) \quad (18)$$

which can be identified by standard method as those of supernormalizable, renormalizable, and non-renormalizable theory respectively (degree of divergence decreasing, constant, and increasing respectively with rising number of vertices in perturbation theory).

In case of super-renormalizability there exists, as in Fig. 10, a straight line \mathcal{A} which separates those contributions to T_2 which are proportional to a positive power of ε from those with a negative one. The terms on \mathcal{A} itself are independent of ε . Each α_k of (17) then appears as an infinite series of the form:

$$\begin{aligned} \alpha_k(\varepsilon) &= \sum_{l=0}^{\infty} \alpha_{kl} \varepsilon^{\beta-\gamma l}, \\ \beta &= \frac{k+1}{M} (M-2), \\ \gamma &= 1 - \frac{2}{N} - \frac{2}{M}. \end{aligned} \quad (19)$$

Our interest lies in the value $\alpha_k(0)$ which could be determined in principle by analytic continuation of the function $\alpha_k(\varepsilon)$ defined by the series (19) in its region of convergence. We make no attempt to tackle the convergence problem. But if (19) converges for all non-zero ε then it must be possible to compute $\alpha_k(\varepsilon)$ for different values of ε from a finite number of terms, and then to extrapolate to $\alpha_k(0)$.

Using the approximate values of $\alpha_k(0)$ calculated in this way for several k , it is not too difficult to extract some information about $T_2(p)$, even if one is restricted in practice to rather small k .

We assume the existence of the LEHMANN-KÄLLÉN spectral representation¹⁵

$$T_2(p) = \int_0^{\infty} d\kappa^2 \frac{b(\kappa^2)}{p^2 - \kappa^2} \quad (20)$$

with the spectral function $b(\kappa^2)$. To compare this formula with (17 a) we write it in the form

$$T_2(p) = - \int_0^{\infty} d\kappa^2 \frac{b(\kappa^2)}{\kappa^2} \sum_{k=0}^{\infty} \left(\frac{p^2}{\kappa^2} \right)^k \quad (20 a)$$

and find¹⁶

$$\alpha_k = - \int_0^{\infty} d\kappa^2 \frac{b(\kappa^2)}{\kappa^{2+2k}}. \quad (21)$$

Our α_k can thus be recognized as moments of the spectral function. Assuming a non-degenerate, discrete mass- (resp. for $N=1$ energy-) spectrum we can determine from the spectral moments α_k the lowest eigenvalues and the corresponding matrix elements. For, we must only take k sufficiently large to suppress the second lowest eigenvalue in order to determine κ_0^2 from

$$\kappa_0^2 = \lim_{k \rightarrow \infty} \frac{\alpha_k}{\alpha_{k+1}} \text{ for } b(\kappa^2) = \sum b_i \delta(\kappa^2 - \kappa_i^2). \quad (22)$$

If a fair number of the α_k is known with sufficient accuracy it is not difficult to go on with the determination of subsequent eigenvalues κ_i^2 .

In the case of a theory on the verge of renormalizability [$M=2N/(N-2)$] the ε -power remains constant and positive throughout each vertical of Fig. 7. The immediate consequence is that T_2 vanishes in the limit $\varepsilon \rightarrow 0$, leading to the statement that the S -matrix is unity (other T_p show similar behaviour as can be seen by generalization of the argument in Appendix D). In order to avoid that result one could combine the ε -power of each vertical with the (p^2) to get

$$T_2(p) = \varepsilon^{\frac{M-2}{2}} \sum \bar{a}_k \left(\varepsilon^{\frac{M-2}{2}} p^2 \right) \sum_i \frac{\bar{b}_i}{p^2 - (\kappa_i/\varepsilon^{1-2/M})} \quad (23)$$

defining renormalized \bar{a}_k by

$$\alpha_k = \bar{a}_k \varepsilon^q. \quad (24)$$

Then it turns out that all eigenvalues can be renormalized with the same constant

$$\varepsilon^{(2-M)/M} \quad (25)$$

such that this constant is only an (unobservable) scale factor.

In the case of non-renormalizable theories, the ε -powers in Fig. 7 remain positive but can no more be united with the (p^2) . Then each eigenvalue is to be renormalized by another constant such that the spectrum becomes essentially ε -dependent.

It seems to demand only minor alterations to treat the case of a general p -point-function. We would like to postpone that generalization as well as numerical calculations until we have incorporated the much more interesting case of FERMION fields. We hope to return to the subject later.

¹⁵ H. LEHMANN, *Nuovo Cim.* **11**, 342 [1954].

¹⁶ The b_i are related to the matrix elements by $b_i = |\langle 0 | \varphi | i \rangle|^2 \kappa_i$.

Acknowledgement

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Appendix A

Relation between the $f_0^{(\nu)}$, M_k , and φ_n

At the one hand, we have

$$\Omega = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n \varphi_n(x_1 \dots x_n) \varrho(x_1) \dots \varrho(x_n) \quad (\text{A } 1)$$

at the other, we expand

$$\begin{aligned} \Omega &= \exp \left\{ \int dx f[\varrho(x)] \right\} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left[\sum_{\nu=0}^{\infty} \frac{1}{\nu!} f_0^{(\nu)} \int dx \varrho^{\nu}(x) \right]^m \\ &= \sum_{m=0}^{\infty} \sum_{\nu_j} \frac{a_1^{\nu_1} a_2^{\nu_2} \dots}{\nu_1! \nu_2! \dots} \end{aligned} \quad (\text{A } 2)$$

with

$$\sum_j \nu_j = m \quad \text{and} \quad a_j = \frac{f_0^{(j)}}{j!} \int dx \varrho^j(x).$$

Comparison yields

$$\begin{aligned} \int dx_1 \dots dx_n \varphi_n(x_1 \dots x_n) \varrho(x_1) \dots \varrho(x_n) \\ = \frac{n!}{i^n} \sum_{\nu_j} \prod_j \frac{1}{\nu_j!} \left[\frac{f_0^{(j)}}{j!} \int dx \varrho^j(x) \right]^{\nu_j} \\ \text{with} \quad \sum_{j=1}^{\infty} j \nu_j = n. \end{aligned} \quad (\text{A } 3)$$

Explicit evaluation of (A 3) leads just to the above (14) mentioned and used structure of the φ_n -functions as sums of products of δ -functions with derivatives of f at the point $\varrho=0$ as factors. For practical calculations we need further the explicit relationship between the M_k from eqs. (11) and (12) and the $f_0^{(\nu)}$. To derive it we put

$$e^{\varepsilon f(\varrho)} \equiv I, \quad (\text{A } 4)$$

then we simply have

$$M_k = \left. \frac{\partial^k I}{\partial \varrho^k} \right|_{\varrho=0}. \quad (\text{A } 5)$$

Let us write

$$\varepsilon f = \ln I = \ln(I_0 + R) = - \sum_{n=1}^{\infty} (-1)^n \frac{R^n}{n}$$

with $R = I - I_0$, $I_0 = I[\varrho=0] = 1$

(per definitionem). Further we notice that

$$R = \sum_{k=1}^{\infty} \frac{\varrho^k}{k!} I^{(k)}(0) = \sum_{k=1}^{\infty} \frac{\varrho^k}{k!} M_k.$$

So we get

$$\varepsilon f(\varrho) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{i_j=n} n! \frac{\left(\frac{\varrho}{1!} M_1 \right)^{i_1} \left(\frac{\varrho^2}{2!} M_2 \right)^{i_2} \dots}{i_1! i_2! \dots}.$$

Comparing this with

$$f(\varrho) = \sum_{\nu} \frac{\varrho^{\nu}}{\nu!} f^{(\nu)}(0)$$

we arrive at

$$f_0^{(\nu)} = \frac{\nu!}{\varepsilon} \sum_{l=1}^{\nu} \sum_{i_j} (-1)^{l-1} (l-1)! \prod_j \frac{1}{i_j!} \left(\frac{M_j}{j!} \right)^{i_j} \quad (\text{A } 6)$$

with $\sum_j i_j = l$, $\sum_j j i_j = \nu$.

That means, e. g. for $\nu=1$: $\varepsilon f_0^{(1)} = M_1$,
and for $\nu=2$: $\varepsilon f_0^{(2)} = M_2 - M_1^2$.

Appendix B

Vacuum effects and T_0

If one collects all terms in the expansion (8) one finds that each of the graphs discussed above is only the simplest one of an infinite series. The first term of this series is represented by a graph without completely separated parts. All other terms correspond to disconnected graphs. Symbolically we can write (Fig. 8)

$$T_2 = T_2^{(c)} (1 + \text{graph} + \text{graph} + \dots) \quad (\text{B } 1)$$

Fig. 8. Equ. B 1.

where $T_2^{(c)}$ contains no disconnected graphs. It is clear that all the contributions in the bracket are simply vacuum effects as they are without any free ends.

Now we remark that T_0 from (8) has the structure

$$T_0 = \varphi_0 + i \int d_{12} G \varphi_2 + \frac{i^2}{2!} d_{1234} G G \varphi_4 + \dots$$

This can be written symbolically in the form (Fig. 9)

$$T_0 = \varphi_0 (1 + \text{graph} + \text{graph} + \dots) \quad (\text{B } 2)$$

Fig. 9. Equ. B 2.

It follows that—apart from a factor— T_0 is exactly the same series as that multiplying $T_0^{(c)}$. Therefore, we can cancel not only φ_0 but all disconnected graphs; they do not contribute to the two point function $\chi = -T_2/T_0$.

Appendix C

Weight factors

To determine the weight of a particular n -th order graph we imagine each of its $n/2-1$ lines endowed with direction and individuality. Then we have to count all turnings and permutations of the lines which result

in new realizations of the given graph.

The weight is given as

$$G = (n-2)!! \frac{2^{-s}}{\prod p_i! \prod g_j!} \quad (C 1)$$

Here $(n-2)!! = 2^{n/2-1}(n/2-1)!$ is the weight of the corresponding basic graph. It arises from the $2^{n/2-1}$ turnings and the $(n/2-1)!$ permutations of the $n/2-1$ single lines.

The factor 2^{-s} in (C 1) accounts for s symmetric loops. A loop is defined as a structure connected to only one point of the residual graph. A symmetric loop can be run without change in either of two directions thus halving the possibilities of new cases.

The factor $\prod p_i!$ arises from the permutations of lines within each p -fold bound.

The factor $\prod g_i!$ takes account of g_i identical loops connected to the same point.

The right combination of $f_0^{(v)}$ -factors for a given graph is easily found by the rule that each k -fold point (resulting from the fusion of $k/2$ double-points) yields one $f_0^{(k)}$.

Appendix D

ε -dependence

Simple dimensional reasoning is sufficient to determine the distribution of the powers of $\varepsilon = \Delta v$ over the scheme Fig. 7. This is so because we have for theories with the Lagrangian (9) in the limit $\rho \rightarrow 0$ only three dimension-bearing quantities, namely \square , ε , and λ . The powers of \square and λ are easily determined for each point of Fig. 7. The abscissa scales already the power k of \square , and from (11), (14) and (A 6) it follows that the index n at the ordinate determines uniquely the λ -dependence of all graphs derived from φ_n :

$$\varphi_n \propto \lambda^{-n/M}. \quad (D 1)$$

To find out the ε -power of a particular point in Fig. 7. We have yet to determine the dimensionalities of the T_2 -function and of λ . To do this, we remember that in our system ($\hbar=c=1$) all quantities take the dimension of a power of length l . Of course, the dimensionalities of δ , \square , and ε are in N -dimensional space-time:

$$[\delta(x)] = l^{-N}, \quad [\square] = l^{-2}, \quad [\varepsilon] = l^N. \quad (D 2)$$

The commutation relation $[\dot{\varphi}(t, r), \varphi(t, r')] = \delta(r - r')$ yields

$$[\varphi] = l^{1-N/2}. \quad (D 3)$$

Now, all terms in (9) must share the same dimensionality. It follows

$$[\lambda] = l^{MN/2-M-N}, \quad [\rho] = l^{-1-N/2}. \quad (D 4)$$

The symbol of functional differentiation has the dimensionality

$$\left[\frac{\delta}{\delta \varphi(x)} \right] = \left[\frac{1}{\varphi} \right] [\delta(x)] = l^{1-N/2} \quad (D 5)$$

so that the two-point-function (1) takes

$$[T_2(x)] = l^{2-N}. \quad (D 6)$$

This suffices to determine the power α of ε at the point (n, k) of Fig. 7 from

$$[T_2(x)] = [\varepsilon^\alpha] [\lambda^{-n/M}] [\square^k] [\delta(x)] \quad (D 7)$$

as

$$\alpha = \frac{2(k+1)}{N} + n \left(\frac{1}{2} - \frac{1}{N} - \frac{1}{M} \right). \quad (D 8)$$

The determination of the ε -powers for a general p -point-function can be accomplished completely analogous. Fig. 10 shows the distribution of ε -powers for the two-point-function in the case $N=1, M=4$.

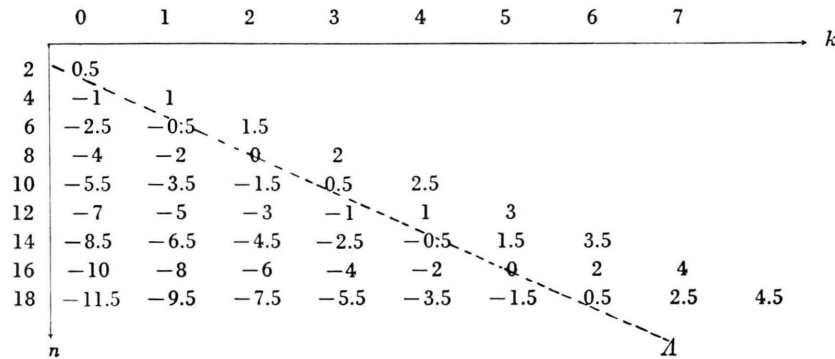


Fig. 10. Distribution of powers of lattice cell volume ε throughout Fig. 7 for case $N=1, M=4$. Dashed line Δ connects ε -independent points.